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Addition theorems and the Drach superintegrable systems

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Abstract
We propose a new construction of the polynomial integrals of motion related to the addition theorems. As an example we reconstruct Drach systems and get some new two-dimensional superintegrable Stäckel systems with third-, fifth- and seventh-order integrals of motion.

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1. Introduction

The Liouville classical theorem on completely integrable Hamiltonian systems implies that almost all points of the manifold M are covered by a system of open toroidal domains with the action–angle coordinates $I = (I_1, \dots, I_k)$ and $\omega = (\omega_1, \dots, \omega_n)$:

$$\{I_j, I_k\} = \{\omega_i, \omega_k\} = 0, \qquad \{I_j, \omega_k\} = \delta_{ij}. \tag{1.1}$$

The independent integrals of motion $H_1, \dots, H_n$ are functions of the independent action variables $I_1, \dots, I_n$ and the corresponding Jacobian is not equal to zero

$$\det \mathbf{J} \neq 0, \qquad \text{where} \qquad \mathbf{J}_{ij} = \frac{\partial H_i(I_1, \dots, I_n)}{\partial I_j}. \tag{1.2}$$

Let us introduce n functions

$$\phi_j = \sum_k (\mathbf{J}^{-1})_{kj}\omega_k, \tag{1.3}$$

such that

$$\{H_i, \phi_j\} = \sum_{k=1}^{n} \mathbf{J}_{ik}(\mathbf{J}^{-1})_{kj} = \delta_{ij}. \tag{1.4}$$

If the Hamiltonian $H = H_1$ then the $(n-1)$ functions $\phi_2, \dots, \phi_n$ are integrals of motion

$$\frac{\mathrm{d}\phi_j}{\mathrm{d}t} = \{H_1, \phi_j\} = 0, \qquad j = 2, \dots, n,$$

which are functionally independent of n functions $H_1(I), \dots, H_n(I)$.

So, in classical mechanics any completely integrable system is a superintegrable system in the neighborhood of any regular point of M [14]. This means that the Hamiltonian $H = H_1$ has $2(n-1)$ integrals of motion $H_2, \dots, H_n$ and $\phi_2, \dots, \phi_n$ on any open toroidal domain.

If the action–angle variables are global variables on the whole phase-space M and, therefore, we have superintegrable systems on M. For instance, the global action–angle variables for the open and periodic Toda lattices are discussed in [3].

However, in generic case the angle variables ω_k are multi-valued functions on the whole phase-space M. If we have k additional single-valued algebraic integrals of motion K, the trajectories are closed (more generally, they are constrained to an $(n-k)$-dimensional manifold in phase-space).

Any additional integral is a function on the action–angle variables. Since we have to understand how to get single-valued additional integrals of motion from the multi-valued action–angle variables. In this paper we discuss a possibility to get polynomial integrals of motion from the multi-valued angle variables by using the simplest addition theorem.

2. The Stäckel systems

The system associated with the name of Stäckel [9, 10] is a holonomic system on the phase-space $M = \mathbb{R}^{2n}$, with the canonical variables $q = (q_1, \dots, q_n)$ and $p = (p_1, \dots, p_n)$:

$$\Omega = \sum_{j=1}^{n} \mathrm{d}p_j \wedge \mathrm{d}q_j, \qquad \{p_j, q_k\} = \delta_{jk}. \tag{2.1}$$

The nondegenerate $n \times n$ Stäckel matrix S, whose j column depends on the coordinate q_j only, defines n functionally independent integrals of motion

$$H_k = \sum_{j=1}^{n} (S^{-1})_{jk} \big(p_j^2 + U_j(q_j)\big). \tag{2.2}$$

From this definition one immediately gets the separated relations

$$p_j^2 = \sum_{k=1}^{n} H_k S_{kj} - U_j(q_j) \tag{2.3}$$

and the angle variables

$$\omega_i = \sum_{j=1}^{n} \int \frac{S_{ij}\,\mathrm{d}q_j}{p_j} = \sum_{j=1}^{n} \int \frac{S_{ij}\,\mathrm{d}q_j}{\sqrt{\sum_{k=1}^{n} H_k S_{kj} - U_j(q_j)}}.$$

It allows reducing the solution of the equations of motion to a problem in the algebraic geometry [10]. Namely, let us suppose that there are functions μ_j and λ_j on the canonical separated variables

$$\mu_j = u_j(q_j) p_j, \qquad \lambda_j = v_j(q_j), \qquad \{q_i, p_j\} = \delta_{ij}, \tag{2.4}$$

which allows us to rewrite separated equations (2.3) as equations defining the hyperelliptic curves

$$\mathcal{C}_j: \quad \mu_j^2 = P_j(\lambda_j) \equiv u_j^2(\lambda_j) \left(\sum_{k=1}^{n} H_k S_{kj}(\lambda_j) - U_j(\lambda_j) \right), \tag{2.5}$$

where $P_j(\lambda_j)$ are polynomials on λ_j. In this case the action variables $I_k = H_k$ (2.2) have the canonical Poisson brackets (1.1) with the angle variables

$$\omega_i = \sum_{j=1}^{n} \int_{A_j} \frac{S_{ij}(\lambda_j)}{\sqrt{P_j(\lambda_j)}} \, \mathrm{d}\lambda_j = \sum_{j=1}^{n} \vartheta_{ij}(p_j, q_j), \tag{2.6}$$

which are the sums of integrals ϑ_{ij} of the first-kind Abelian differentials on the hyperelliptic curves $\mathcal{C}_j$ (2.5) [10, 14], i.e. they are sums of the multi-valued functions on the whole phase-space.

2.1. *Addition theorems and algebraically superintegrable systems*

In the generic case the action variables (2.6) are the sum of the multi-valued functions ϑ_{ij}. However, if we are able to apply some addition theorem to the calculation of ω_i (2.6)

$$\omega_i = \sum_{j=1}^{n} \vartheta_{ij}(p_j, q_j) = \Theta_i(K_i) + \text{const}, \tag{2.7}$$

where Θ_i is a multi-valued function on the algebraic argument $K_i(p, q)$, then one will get algebraic integrals of motion $K_i(p, q)$ because

$$\{H_1, \omega_i\} = \{H_1, \Theta_i(K_i)\} = \Theta_i' \cdot \{H_1, K_i\} = 0.$$

So, the addition theorems (2.7) could help us to classify algebraically superintegrable systems and vice versa. Plane curves with the genus $g \geqslant 1$ are related to elliptic and Abelian integrals. Addition theorems of these functions are the content of Abel's theorem [15].

The main result of this paper is that almost all the known examples of algebraically superintegrable systems relate with the one of the simplest addition theorem

$$\mathrm{e}^x \mathrm{e}^y = \mathrm{e}^{x+y}, \qquad \text{or} \qquad \ln(x_1) + \ln(x_2) = \ln(x_1 x_2) \tag{2.8}$$

associated with the zero-genus hyperelliptic curves $\mathcal{C}_j$

$$\mathcal{C}_j: \quad \mu_j^2 = P_j(\lambda_j) = f_j \lambda_j^2 + g_j \lambda_j + h_j, \qquad j = 1, \ldots, n, \tag{2.9}$$

where f_j, g_j, h_j are linear functions on n integrals of motion $H_1, \ldots, H_n$.

In fact, if $S_{ij}(\lambda) = 1$ then after substituting (2.9) into (2.6) one gets the sum of the rational functions

$$\vartheta_j = \int \frac{1}{\sqrt{g_j \lambda_j + h_j}} \, \mathrm{d}\lambda_j = \frac{\mu_j}{g_j}$$

or the logarithmic functions

$$\vartheta_j = \int \frac{1}{\sqrt{f_j \lambda_j^2 + g_j \lambda_j + h_j}} \, \mathrm{d}\lambda = f_j^{-1/2} \ln \left(\mu_j + \frac{2 f_j \lambda_j + g_j}{2\sqrt{f_j}} \right).$$

For the hyperelliptic curves of higher genus one gets elliptic functions, which have a more complicated addition law [15].

In order to use the addition law (2.8) we have to make the following steps:

- We have to apply canonical transformation of the time that reduce the nth row of the Stäckel matrix to the canonical Brill–Noether form [11, 13]

  $$S_{nj} = 1, \qquad j = 1, \ldots n,$$

  such that

  $$\omega_n = \frac{1}{N} \sum_{j=1}^{n} \int^{v_j(q_j)} \frac{1}{\sqrt{f_j \lambda^2 + g_j \lambda + h_j}} \, \mathrm{d}\lambda, \tag{2.10}$$

  where N is normalization, which is restored from $\{\omega_n, H_n\} = 1$.

- Then we have to use the addition law (2.8) for the construction of the polynomial in momenta integrals of motion.
- We have to make an inverse transformation of the time, which preserves the polynomial form of integrals such that it depends on q variables only [11, 13].

Let us consider the construction of the polynomial in the momenta integrals of motion at $n = 2$.

2.1.1. Case $f_1 = f_2 = 0$. If $f_{1,2} = 0$, we have

$$\omega_2 = \frac{1}{4}\sum_{j=1}^{2}\int^{v_j(q_j)} \frac{1}{\sqrt{g_j\lambda + h_j}}\,\mathrm{d}\lambda = \frac{p_1u_1}{g_1} + \frac{p_2u_2}{g_2} = \frac{K}{g_1g_2}, \tag{2.11}$$

where

$$K = g_2p_1u_1 + g_1p_2u_2$$

is the polynomial in the momenta integral of motion of the first or the third order. Recall that $g_{1,2}$ are linear functions on the Stäckel integrals $H_{1,2}$, which are the second-order polynomials on $p_{1,2}$.

Example 1. Let us consider the two-dimensional Stäckel system defined by two Riemann surfaces

$$\mathcal{C}_{1,2}: \quad \mu^2 = P_{1,2}(\lambda) = (H_1 \pm H_2)\lambda + \alpha_{1,2},$$

and substitutions (2.4)

$$\mu_j = q_jp_j, \qquad \lambda_j = q_j^2,$$

which give rise to the following separated equations:

$$p_{1,2}^2 = H_1 \pm H_2 + \frac{\alpha_{1,2}}{q_{1,2}^2}.$$

Integrals of motion H_1 and H_2 are solutions of these separated equations

$$H_{1,2} = \frac{p_1^2 \pm p_2^2}{2} - \frac{\alpha_1}{2q_1^2} \mp \frac{\alpha_2}{2q_2^2}, \tag{2.12}$$

which coincide with integrals of motion for the two-particle Calogero system after an obvious point transformation

$$x = \frac{q_1 - q_2}{2}, \qquad p_x = p_1 - p_2, \qquad y = \frac{q_1 + q_2}{2}, \qquad p_y = p_1 + p_2. \tag{2.13}$$

The angle variables (2.11) read as

$$\begin{aligned}\omega_{1,2} &= \frac{1}{4}\int^{q_1^2} \frac{1}{\sqrt{(H_1 + H_2)\lambda + \alpha_1}}\,\mathrm{d}\lambda \pm \frac{1}{4}\int^{q_2^2} \frac{1}{\sqrt{(H_1 - H_2)\lambda + \alpha_2}}\,\mathrm{d}\lambda\\ &= \frac{-p_1q_1(H_1 - H_2) \mp p_2q_2(H_1 + H_2)}{2(H_1 + H_2)(H_1 - H_2)}.\end{aligned}$$

The corresponding cubic integral of motion K (2.11) is equal to

$$K = 2(H_1 + H_2)(H_1 - H_2)\omega_2 = -p_1q_1(H_1 - H_2) + p_2q_2(H_1 + H_2).$$

The bracket

$$\{K, H_2\} = 2\left(H_2^2 - H_1^2\right)$$

is easily restored from the canonical brackets (1.1).

2.1.2. Case $f_1 = k_1^2 f$ and $f_2 = k_2^2 f$ with integer $k_{1,2}$. In this case, we have

$$\omega_2 = \frac{1}{2}\sum_{j=1}^{2}\int^{v_j(q_j)} \frac{1}{\sqrt{f_j\lambda^2 + g_j\lambda + h_j}}\,\mathrm{d}\lambda = \sum_{j=1}^{2}\frac{1}{2k_j\sqrt{f}}\ln\left(\frac{P_j'}{2k_j\sqrt{f}} + p_ju_j\right)$$
$$= \frac{1}{2k_1k_2\sqrt{f}}\ln\left[\left(\frac{P_1'}{2k_1\sqrt{f}} + p_1u_1\right)^{k_2}\left(\frac{P_2'}{2k_2\sqrt{f}} + p_2u_2\right)^{k_1}\right],$$

where

$$P_j' = \left.\frac{\mathrm{d}P_j(\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=v_j(q_j)} = 2f_jv_j(q_j) + g_j.$$

So function

$$\Phi_2 = \exp(2k_1k_2\sqrt{f}\omega_2) = \left(\frac{P_1'}{2k_1\sqrt{f}} + p_1u_1\right)^{k_2}\left(\frac{P_2'}{2k_2\sqrt{f}} + p_2u_2\right)^{k_1}$$
$$= \frac{1}{f^{(k_1+k_2)/2}}(K_\ell + \sqrt{f}K_m) \tag{2.14}$$

is the generating function of polynomial integrals of motion K_m and K_ℓ of the mth and ℓth order, respectively. The values of m and ℓ depend on the values of $k_{1,2}$.

As above, the algebra of integrals $H_{1,2}$ and K_m may be restored from the canonical brackets (1.1).

Example 2. Let us consider the two-dimensional Stäckel system defined by the two Riemann surfaces

$$\mathcal{C}_{1,2}: \quad \mu_{1,2}^2 = P_{1,2}(\lambda_{1,2}) = k_{1,2}^2\lambda_{1,2}^2 + \beta_{1,2}\lambda_{1,2} + (H_1 \pm H_2)$$

and substitutions (2.4)

$$\mu_j = p_j, \qquad \lambda_j = q_j,$$

which give rise to the following separated equations:

$$p_{1,2}^2 = k_{1,2}^2q_{1,2}^2 + \beta_{1,2}q_{1,2} + (H_1 \pm H_2).$$

Integrals of motion H_1 and H_2 are solutions of these separated equations

$$H_{1,2} = \frac{p_1^2 \pm p_2^2}{2} - \frac{k_1^2q_1^2 \pm k_2^2q_2^2}{2} - \frac{\beta_1q_1 \pm \beta_2q_2}{2}, \tag{2.15}$$

which coincide with the integrals of motion for the harmonic oscillator. The same Stäckel system coincides with the Kepler problem after a well-known canonical transformation of the time, which changes the row of the Stäckel matrix [11].

The angle variable (2.11) reads as

$$\omega_2 = -\frac{1}{2}\left(\int^{q_j}\frac{1}{\sqrt{k_1^2\lambda^2 + \beta_1\lambda + (H_1 + H_2)}}\,\mathrm{d}\lambda - \int^{q_2}\frac{1}{\sqrt{k_2^2\lambda^2 + \beta_2\lambda + (H_1 - H_2)}}\,\mathrm{d}\lambda\right)$$
$$= \frac{1}{2\sqrt{k_1^2k_2^2}}\left[\sqrt{k_2^2}\ln\left(p_1 + \frac{P_1'}{2\sqrt{k_1^2}}\right) - \sqrt{k_1^2}\ln\left(p_2 + \frac{P_2'}{2\sqrt{k_2^2}}\right)\right].$$

Using suitable branches of $\sqrt{k_1^2}$, one gets function (2.14)

$$\Phi = \exp(2k_1k_2\omega_2) = \left(p_1 - \frac{2k_1^2q_1 + \beta_1}{2k_1}\right)^{k_2}\left(p_2 + \frac{2k_2^2q_2 + \beta_2}{2k_2}\right)^{k_1},$$

which generates the polynomial integrals of motion K_m and K_ℓ. As an example, if $k_1 = 1$ and $k_2 = 3$ one gets the third- and fourth-order integrals of motion K_m and K_ℓ, respectively.

Of course, the oscillator is one of the well-studied superintegrable systems which was earmarked to illustrate generic construction only. The addition theorem (2.8) allows us to get a huge family of the n-dimensional superintegrable systems, which have to be classified and studied.

2.2. Classification

In order to classify superintegrable systems associated with the addition theorem (2.8), we have to start with a pair of the Riemann surfaces

$$\mathcal{C}_j: \quad \mu^2 = P_j(\lambda) = f\lambda^2 + g_j\lambda + h_j, \qquad j = 1, 2, \tag{2.16}$$

where

$$f = \alpha H_1 + \beta H_2 + \gamma, \qquad g_j = \alpha_j^g H_1 + \beta_j^g H_2 + \gamma_j^g, \qquad h_j = \alpha_j^h H_1 + \beta_j^h H_2 + \gamma_j^h,$$

and α, β and γ are real or complex numbers.

In order to use the addition theorem (2.8) we fix the last row of the Stäckel matrix. Namely, substituting

$$S_{2j}(\lambda) = \kappa_j \tag{2.17}$$

into (2.3)–(2.6) one gets

$$\vartheta_{2j} = \int \frac{S_{2j}(q_j)\,\mathrm{d}q_j}{p_j} = \int \frac{\kappa_j\,\mathrm{d}\lambda_j}{\sqrt{P_j}} = \kappa_j f^{-1/2} \ln\left(\mu_j + \frac{2f\lambda_j + g_j}{2\sqrt{f}}\right), \tag{2.18}$$

so the angle variable

$$\omega_2 = \frac{1}{\sqrt{f}} \ln\left[\left(p_1 u_1 + \frac{P_1'}{2\sqrt{f}}\right)^{\kappa_1}\left(p_2 u_2 + \frac{P_2'}{2\sqrt{f}}\right)^{\kappa_2}\right], \qquad P_j' = \left.\frac{\mathrm{d}P_j(\lambda)}{\mathrm{d}\lambda}\right|_{\lambda = v_j(q_j)}$$

is the multi-valued function on the desired algebraic argument

$$K = \left(p_1 u_1 + \frac{P_1'}{2\sqrt{f}}\right)^{\kappa_1}\left(p_2 u_2 + \frac{P_2'}{2\sqrt{f}}\right)^{\kappa_2}.$$

If $\kappa_{1,2}$ are positive integer, then

$$K = \left(\frac{1}{2\sqrt{f}}\right)^{\kappa_1+\kappa_2}\left(K_\ell + \sqrt{f}K_m\right) \tag{2.19}$$

is the generating function of polynomial integrals of motion K_m and K_ℓ of the mth and $(m \pm 1)$th order in the momenta, respectively.

As an example, we have

$$\begin{aligned} K_m &= 2(p_1u_1P_2' + p_2u_2P_1'), && \kappa_1 = 1, \kappa_2 = 1,\\ K_m &= 2P_2'(2p_2u_2P_1' + p_1u_1P_2') + 8p_1u_1p_2^2u_2^2 f, && \kappa_1 = 1, \kappa_2 = 2,\\ K_m &= 2P_2'^2(3p_2u_2P_1' + p_1u_1P_2') + 8p_2^2u_2^2(p_2u_2P_1' + 3p_1u_1P_2')f, && \kappa_1 = 1, \kappa_2 = 3, \end{aligned} \tag{2.20}$$

where $m = 1, 3$, $m = 3, 5$ and $m = 3, 7$, because $P_{1,2}'$ and f are linear functions on $H_{1,2}$, which are the second-order polynomials on momenta.

The corresponding expressions for the K_ℓ look like

$$\begin{aligned} K_\ell &= P_1' P_2' + 4 p_1 p_2 u_1 u_2 f, && \kappa_1 = 1, \kappa_2 = 1, \\ K_\ell &= P_1' P_2'^2 + 4 f p_2 u_2 (p_2 u_2 P_1' + 2 p_1 u_1 P_2'), && \kappa_1 = 1, \kappa_2 = 2, \\ K_\ell &= P_1' P_2'^3 + 12 P_2' p_2 u_2 (P_2' p_1 u_1 + P_1' p_2 u_2) f + 16 p_1 u_1 p_2^3 u_2^3 f^2, && \kappa_1 = 1, \kappa_2 = 3. \end{aligned} \tag{2.21}$$

The imposed condition (2.17) leads to some restrictions on the functions $v_j(q_j)$ and $u_j(q_j)$. In fact, substituting canonical variables (2.4) into equations (2.16) we obtain the following expression for the Stäckel matrix:

$$S = \begin{pmatrix} \dfrac{\alpha v_1^2 + \alpha_1^g v_1 + \alpha_1^h}{u_1^2} & \dfrac{\alpha v_2^2 + \alpha_2^g v_2 + \alpha_2^h}{u_2^2} \\ \dfrac{\beta v_1^2 + \beta_1^g v_1 + \beta_1^h}{u_1^2} & \dfrac{\beta v_2^2 + \beta_2^g v_2 + \beta_2^h}{u_2^2} \end{pmatrix}, \qquad \det S \neq 0. \tag{2.22}$$

So, the given $\kappa_{1,2}$ expressions for ϑ_{2j} (2.18) yield two differential equations on functions u, v and parameters β:

$$S_{2j}(q_j) = \frac{\kappa_j v_j'(q_j)}{u_j(q_j)} \quad \Longrightarrow \quad \kappa_j u_j v_j' = \beta v_j^2 + \beta_j^g v_j + \beta_j^h, \qquad j = 1, 2. \tag{2.23}$$

For the Stäckel systems with rational or trigonometric metrics, we have to solve these equations in the space of the truncated Laurent or Fourier polynomials, respectively.

Proposition 1. *If $\kappa_j \neq 0$, equations (2.23) have the following three monomial solutions:*

$$\begin{array}{lllll} \mathrm{I} & \beta = 0, & \beta_j^h = 0, & u_j = q_j, & v_j = q_j^{\frac{\beta_j^g}{\kappa_j}}, \\ \mathrm{II} & \beta_j^g = 0, & \beta_j^h = 0, & u_j = 1, & v_j = -\kappa_j (\beta q_j)^{-1}, \\ \mathrm{III} & \beta = 0, & \beta_j^g = 0, & u_j = 1, & v_j = \kappa_j^{-1} \beta_j^h q_j, \end{array} \tag{2.24}$$

up to canonical transformations. The fourth solution (IV) *is the combination of the first and third solutions for different j's.*

In order to prove this fact, we can substitute $u = aq^m$ and $v = bq^k$ into (2.23) and divide the resulting equation by q^{m+k-1}:

$$abk\kappa = b^2 \beta q^{k+1-m} + q^{1-m} b \beta^g + q^{1-k-m} \beta^h.$$

Finally, we differentiate it w.r.t. q and multiply by q^m:

$$0 = -(m - k - 1) b^2 \beta q^k - (m - 1) b \beta^g - (m + k - 1) \beta^h q^{-k}.$$

As $b \neq 0$ and $k \neq 0$ one gets three solutions (2.24) only.

Then we suppose that after some point transformation

$$\begin{aligned} x &= z_1(q), & y &= z_2(q), \\ p_x &= \mathrm{w}_{11}(q) p_1 + \mathrm{w}_{12}(q) p_2, & p_y &= \mathrm{w}_{21}(q) p_1 + \mathrm{w}_{22}(q) p_2, \end{aligned} \tag{2.25}$$

where $\mathrm{w}_{ij} \neq 0$, kinetic part of the Hamilton function $H_1 = T + V$ has a special form

$$T = \sum (S^{-1})_{1j} p_j^2 = \mathrm{g}_{11}(x, y) p_x^2 + \mathrm{g}_{12}(x, y) p_x p_y + \mathrm{g}_{22}(x, y) p_y^2,$$

where g is a metric on a configurational manifold. For instance, if we suppose that

$$T = \sum (S^{-1})_{1j} p_j^2 = p_x p_y,$$

then one gets the following algebraic equations

$$\mathrm{w}_{11}\mathrm{w}_{21} = (S^{-1})_{11}, \qquad \mathrm{w}_{12}\mathrm{w}_{21} + \mathrm{w}_{11}\mathrm{w}_{22} = 0, \qquad \mathrm{w}_{12}\mathrm{w}_{22} = (S^{-1})_{12} \tag{2.26}$$

and the partial differential equations

$$\{x, p_x\} = \{y, p_y\} = 1, \qquad \{p_x, y\} = \{p_y, x\} = \{p_x, p_y\} = 0 \tag{2.27}$$

on parameters α and functions $z_{1,2}(q_1, q_2)$ and $\mathrm{w}_{kj}(q_1, q_2)$.

The remaining free parameters $\gamma, \gamma_j^h, \gamma_j^g$ determine the corresponding potential part of the Hamiltonian $V(x, y)$. In fact, since integrals $H_{1,2}$ are defined up to the trivial shifts $H_k \to H_k + c_k$, our potential $V(x, y)$ depends on three arbitrary parameters only.

Summing up, in order to get all the superintegrable systems on a complex Euclidean space $E_2(\mathbb{C})$ associated with the addition theorem (2.8), we have to solve equations (2.23), (2.26) and (2.27) with respect to functions $u_j(q_j)$, $v_j(q_j)$, $z_{1,2}(q_1, q_2)$ and $\mathrm{w}_{kj}(q_1, q_2)$ and parameters α and β.

Example 3. Let us consider the second solution from list (2.24) at $\kappa_1 = 1$ and $\kappa_2 = 2$. In this case, equations (2.26) and (2.27) have the following partial solution:

$$S = \begin{pmatrix} 1 & 0 \\ \dfrac{1}{q_1^2} & \dfrac{4}{q_2^2} \end{pmatrix}, \qquad \begin{aligned} x &= \sqrt{q_1} q_2, & p_x &= \frac{\sqrt{q_1}}{q_2} p_1 + \frac{1}{2\sqrt{q_1}} p_2, \\ y &= \frac{\sqrt{q_1}}{q_2}, & p_y &= \sqrt{q_1} q_2 p_1 - \frac{q_2^2}{2\sqrt{q_1}} p_2. \end{aligned}$$

Adding potential terms one gets two Riemann surfaces:

$$\mathcal{C}_1 : \quad \mu^2 = H_2\lambda^2 - H_1\lambda - \gamma_1 \qquad \text{and} \qquad \mathcal{C}_2 : \quad \mu^2 = H_2\lambda^2 - 2\gamma_2\lambda + 4\gamma_3,$$

where $\mu_j = p_j, \lambda_j = q_j^{-1}$. Solutions of the corresponding separated variables are the second-order Stäckel integrals of motion, which in physical variables look like

$$H_1 = p_x p_y + \gamma_1 xy + \frac{\gamma_2}{\sqrt{xy^3}} + \frac{\gamma_3}{y^2}, \qquad H_2 = \frac{(p_x x - p_y y)^2}{4} - \gamma_2\sqrt{\frac{x}{y}} - \gamma_3 \frac{x}{y}. \tag{2.28}$$

It is a new integrable system, which is missed in the known lists of superintegrable systems [1, 6–8].

In this case, the integrals of motion K_5 (2.20) and K_6 (2.21) are the fifth- and sixth-order polynomials in the momenta, respectively. Of course, we can try to get the quartic, cubic and quadratic integrals of motion K_4, K_3 and K_2, respectively, from the recurrence relations

$$K_5 = \{K_4, H_2\}, \qquad K_4 = \{K_3, H_2\}, \qquad K_3 = \{K_2, H_2\} \tag{2.29}$$

and the equation $\{H_1, K_j\} = 0$, $j = 4, 3, 2$.

Solving the first recurrence equation, one gets quartic integral of motion

$$K_4 = -4(xp_x - yp_y)^2\left(p_x^2 + \gamma_1 y^2\right) - \frac{4\gamma_2^2}{xy} + \frac{8p_x(xp_x - yp_y)\gamma_2}{\sqrt{xy}} + \frac{16x\left(p_x^2 + \gamma_1 y^2\right)\gamma_3}{y}.$$

The other recurrence relations (2.29) have no polynomial solutions. Of course, we can try to modify recurrence relations

$$\{K_3, H_2\} = K_4 + F_4(H_1, H_2), \qquad \{K_2, H_2\} = K_3 + F_3(H_1, H_2)$$

in order to get cubic and quadratic integrals of motion. However, using ansatz

$$K_2 = h_1(x,y)p_x^2 + h_2(x,y)p_x p_y + h_3(x,y)p_y^2 + h_4(x,y)$$

we can directly prove that there is no additional quadratic integral of motion, which commute with H_1 (2.28).

3. The Drach systems

In 1935 Jules Drach published two articles on the Hamiltonian systems with the third-order integrals of motion on a complex Euclidean space $E_2(\mathbb{C})$ with the following Hamilton function [2]:

$$H_1 = p_x p_y + U(x,y). \tag{3.1}$$

Up to canonical transformations $x \to ax$ and $y \to by$, the corresponding potentials look like [7, 12]

$$(a)\qquad U = \frac{\alpha}{xy} + \beta x^{r_1} y^{r_2} + \gamma x^{r_2} y^{r_1}, \qquad \text{where} \quad r_j^2 + 3r_j + 3 = 0,$$

$$(b)\qquad U = \frac{\alpha}{\sqrt{xy}} + \frac{\beta}{(y-x)^2} + \frac{\gamma(y+x)}{\sqrt{xy}(y-x)^2},$$

$$(c)\qquad U = \alpha xy + \frac{\beta}{(y-x)^2} + \frac{\gamma}{(y+x)^2},$$

$$(d)\qquad U = \frac{\alpha}{\sqrt{y(x-1)}} + \frac{\beta}{\sqrt{y(x+1)}} + \frac{\gamma x}{\sqrt{x^2-1^2}},$$

$$(e)\qquad U = \frac{\alpha}{\sqrt{xy}} + \frac{\beta}{\sqrt{x}} + \frac{\gamma}{\sqrt{y}},$$

$$(f)\qquad U = \alpha xy + \beta y\frac{2x^2+1}{\sqrt{x^2+1}} + \frac{\gamma x}{\sqrt{x^2+1}},$$

$$(g)\qquad U = \frac{\alpha}{(y+x)^2} + \beta(y-x) + \frac{\gamma(3y-x)(y-3x)}{3},$$

$$(h)\qquad U = \left(y + \frac{mx}{3}\right)^{-2/3}\left[\alpha + \beta(y - mx/3) + \gamma\left(y^2 - \frac{14mxy}{3} + \frac{m^2x^2}{9}\right)\right],$$

$$(k)\qquad U = \alpha y^{-1/2} + \beta x y^{-1/2} + \gamma x,$$

$$(l)\qquad U = \alpha\left(y - \frac{\rho x}{3}\right) + \beta x^{-1/2} + \gamma x^{-1/2}(y - \rho x).$$

As Drach made some assumptions on the form of the third-order integrals of motion K_3 (3.6) in the calculation it is not immediately clear whether the obtained list is complete.

3.1. Non-separable systems

Let us discuss the Drach systems, which cannot be reduced to the Stäckel systems by any point transformation of variables.

The first system (a) is the non-Stäckel system related to the three-particle periodic Toda lattice in the center-of-mass frame [12] and there are global action–angle variables [3].

The (h) system is reduced to the Stäckel system by a non-point canonical transformation and, therefore, the existence of the third-order integral of motion is related with this non-point transformation [12]. Later, this system has been rediscovered by Holt [4].

For the (k) case in the Drach papers [2], we can find the Hamiltonian

$$H_1^{(k)} = p_x p_y + \alpha y^{-3/2} + \beta x y^{-3/2} + \gamma x \tag{3.2}$$

and the following cubic integral of motion

$$K_3^{(k)} = 6w(x, y)\left(\frac{\partial H}{\partial x} p_y - p_x \frac{\partial H}{\partial y}\right) - P(p_x, p_y, x, y), \tag{3.3}$$

where

$$P = 3p_x^2 p_y, \qquad w = -y.$$

It is easy to prove that $\left\{H_1^{(k)}, K_3^{(k)}\right\} \neq 0$ and, therefore, we have to suggest the possibility of a small mistake in the Drach papers [2].

Following [12], if we solve equation $\left\{p_x p_y + U(x, y), K_3^{(k)}\right\} = 0$ with respect to $U(x, y)$, then one gets our case (k)

$$H_1 = p_x p_y + \alpha y^{-1/2} + \beta x y^{-1/2} + \gamma x.$$

On the other hand, we have directly proved that the Hamiltonian $H_1^{(k)}$ (3.2) has one second-order integral of motion

$$H_2^{(k)} = p_x^2 - 4\beta y^{1/2} + 2\gamma y \tag{3.4}$$

and has not a cubic integral of motion. Moreover, it is easy to see that quadratic integrals of motion $H_{1,2}^{(k)}$ (3.2) and (3.4) cannot be reduced to the Stäckel integrals by any point transformation of variables.

Below we do not consider (a) and (h) systems and consider (k) case in our notation only.

3.2. Classification

For the Drach systems $\kappa_1 = \kappa_2 = \pm 1, 1/2$ and

$$\begin{aligned} \Phi_2 = \exp(2\sqrt{f}\omega_2) &= \left(\frac{P_1'}{2\sqrt{f}} + p_1 u_1\right)\left(\frac{P_2'}{2\sqrt{f}} + p_2 u_2\right) \\ &= \frac{1}{4f}(K_\ell + \sqrt{f} K_m) \end{aligned} \tag{3.5}$$

may be considered as the generating function of the polynomial integrals of motion (2.20) and (2.21)

$$\begin{aligned} K_m &= 2(p_1 u_1 P_2' + p_2 u_2 P_1'), \qquad & m = 1, 3, \\ K_\ell &= P_1' P_2' + 4 p_1 p_2 u_1 u_2 f, \qquad & \ell = 2, 4, \end{aligned} \tag{3.6}$$

of the mth and ℓth order, respectively. It is clear that $m = 1, 3$ and $\ell = 2, 4$, because $P_{1,2}'$ and f are linear functions on $H_{1,2}$, which are second-order polynomials on momenta.

We have to underline that we use different $\kappa_1 = \kappa_2 = \pm 1, 1/2$ for the agreement of K_m (3.6) with the initial Drach integrals of motion [2] only.

One gets the third-order polynomial integral of motion K_3 (3.6) if and only if $P_1'(\lambda)$ or $P_2'(\lambda)$ depends on H_1 or H_2. It leads to the additional restrictions on α and β

$$\sum_{k=1}^{2} \frac{\partial^2 P_j(\lambda)}{\partial H_k \partial \lambda} \neq 0 \qquad \text{for} \quad j = 1 \quad \text{or} \quad j = 2. \tag{3.7}$$

In order to get all the superintegrable systems on a complex Euclidean space $E_2(\mathbb{C})$ associated with the addition theorem (2.8), we have to solve equations (2.23), (2.26), (2.27) and (3.7) at $\kappa_{1,2} = 1$ with respect to the functions $u_j(q_j)$, $v_j(q_j)$, $z_{1,2}(q_1, q_2)$ and $\mathrm{w}_{kj}(q_1, q_2)$ and parameters α and β.

Proposition 2. *The Drach list of the Stäckel systems with the cubic integral of motion (3.6) associated with the addition theorem (2.8) is complete up to canonical transformations of the extended phase-space.*

The results of corresponding calculations may be joined into the following table:

	$\mathcal{C}_{1,2}$ (2.16)	Subs. (2.4)	$z_{1,2}$ (2.25)	S
b	$\mu^2 = H_1\lambda^2 + (H_2 + 2\alpha)\lambda - \beta + 2\gamma$ $\mu^2 = H_1\lambda^2 + (H_2 - 2\alpha)\lambda - \beta - 2\gamma$	$\mu_j = p_j q_j$	$z_{1,2} = \frac{(q_1 \pm q_2)^2}{4}$	$\begin{pmatrix} q_1^2 & q_2^2 \\ 1 & 1 \end{pmatrix}$
c	$\mu^2 = \frac{\alpha}{4}\lambda^2 + \left(H_2 + \frac{H_1}{2}\right)\lambda + \gamma$ $\mu^2 = \frac{\alpha}{4}\lambda^2 + \left(H_2 - \frac{H_1}{2}\right)\lambda - \beta$	$\lambda_j = q_j^2$ $\kappa_j = \frac{1}{2}$	$z_{1,2} = \frac{q_1 \pm q_2}{2}$	$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 \end{pmatrix}$
d	$\mu^2 = H_2\lambda^2 - \sqrt{8}(\alpha + \beta)\lambda + H_1 - 2\gamma$ $\mu^2 = H_2\lambda^2 - \sqrt{8}(\alpha - \beta)\lambda + H_1 + 2\gamma$	$\mu_j = p_j$	$z_1 = \frac{q_1^2 + q_2^2}{2q_1q_2}$ $z_2 = q_1q_2$	$\begin{pmatrix} 1 & 1 \\ \frac{1}{q_1^2} & \frac{1}{q_2^2} \end{pmatrix}$
f	$\mu^2 = H_2\lambda^2 - \left(\frac{\gamma}{2} - H_1\right)\lambda - \frac{\alpha}{4} - \frac{\beta}{2}$ $\mu^2 = H_2\lambda^2 - \left(\frac{\gamma}{2} + H_1\right)\lambda - \frac{\alpha}{4} + \frac{\beta}{2}$	$\lambda_j = q_j^{-1}$ $\kappa_j = -1$	$z_1 = \frac{q_1 - q_2^2}{2\sqrt{q_1q_2}}$ $z_2 = \sqrt{q_1q_2}$	$\begin{pmatrix} \frac{1}{q_1} & \frac{-1}{q_2} \\ \frac{1}{q_1^2} & \frac{1}{q_2^2} \end{pmatrix}$
e	$\mu^2 = H_1\lambda^2 + 2(\beta + \gamma)\lambda + H_2 + 2\alpha$ $\mu^2 = H_1\lambda^2 - 2(\beta - \gamma)\lambda + H_2 - 2\alpha$	$\mu_j = p_j$	$z_{1,2} = \frac{(q_1 \pm q_2)^2}{4}$	$\begin{pmatrix} q_1^2 & q_2^2 \\ 1 & 1 \end{pmatrix}$
k	$\mu^2 = \frac{\gamma}{2}\lambda^2 + (\beta + H_1)\lambda + H_2 + \alpha$ $\mu^2 = \frac{\gamma}{2}\lambda^2 + (\beta - H_1)\lambda + H_2 - \alpha$	$\lambda_j = q_j$ $\kappa_j = 1$	$z_1 = \frac{q_1 - q_2}{2}$ $z_2 = \frac{(q_1 + q_2)^2}{4}$	$\begin{pmatrix} q_1 & -q_2 \\ 1 & 1 \end{pmatrix}$
g	$\mu^2 = -\frac{\gamma}{3}\lambda^2 + \left(\frac{H_1}{2} + H_2\right)\lambda + \alpha$ $\mu^2 = -\frac{\gamma}{3}\lambda^2 - \frac{\beta}{4}\lambda + \frac{H_2}{4} - \frac{H_1}{8}$	$\mu_1 = p_1q_1, \lambda_1 = q_1^2$ $\mu_2 = 2p_1, \lambda_2 = q_2$ $\kappa_j = \frac{1}{2}$	$z_{1,2} = \frac{q_1 \pm q_2}{2}$	$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 \end{pmatrix}$

Similar to the oscillator and the Kepler problem, the Kepler change of the time $t \to \widetilde{t}$, where

$$\mathrm{d}\widetilde{t} = v(q)\,\mathrm{d}t, \qquad v(q) = \frac{\det S}{\det \widetilde{S}} \tag{3.8}$$

relates the Drach systems (*b*), (*d*) and (*e*) with the systems (*c*), (*f*) and (*k*), respectively. Here, S are the Stäckel matrices for (*b*), (*d*) and (*e*) systems and $\widetilde{S}$ are the Stäckel matrices for (*c*), (*f*) and (*k*) systems. Recall that these matrices S and $\widetilde{S}$ have different first row only, see [11, 13].

3.3. Integrals of motion

Using definitions (3.6) we can prove that the integral of motion K_4 (3.6) is the function on H_1, H_2 and K_3:

$$K_4^2 = 16h_1h_2f^2 + \left(K_3^2 - 4h_1g_2^2 - 4h_2g_1^2\right)f + g_1^2g_2^2. \tag{3.9}$$

Substituting this expression into the definition of Φ_2 (3.5), we can get the integral K_3 as a function on the action–angle variables $I_{1,2} = H_{1,2}$ and ω_2.

As usual, polynomial algebra of integrals of motion $H_{1,2}$ and K_3 follows from the canonical brackets (1.1)

$$\begin{aligned}
&\{H_1, H_2\} = \{H_1, K_3\} = \{H_1, K_4\} = 0, \\
&\{H_2, K_3\} = \delta K_4, \qquad \{H_2, K_4\} = \delta f K_3, \\
&\{K_3, K_4\} = F_Z(H_1, H_2, K_3),
\end{aligned} \tag{3.10}$$

where

- $\delta = 4$, $F_I = 16f(g_1h_2 + g_2h_1) - 4g_1g_2(g_1 + g_2)$ for b,c cases;
- $\delta = -2$, $F_{II} = K_3 + 32fh_1h_2 - 4\big(g_1^2h_2 + g_2^2h_1\big)$ for d,f cases;
- $\delta = 2$, $F_{III} = 4f\big(g_1^2 + g_2^2\big) - 16f^2(h_1 + h_2)$ for e,k cases;
- $\delta = 4$, $F_{IV} = 2f\big(g_1^2 + 8g_1h_2\big) - 8f^2h_1 - 4g_1g_2^2$ for g case.

The difference in the values of δ is related with the difference in κs which has been defined by the Drach integrals of motion [2]. Of course, we can put $\kappa_1 = \kappa_2 = 1$ in all the cases such that $\delta = 2$ and polynomials F look as in (4.1). It reduces polynomials $P_{1,2}$ in the table only.

As above (2.29), we can try to find another second-order polynomial integral of motion K_2 from the equations

$$K_3 = \{H_2, K_2\}, \qquad \{H_1, K_2\} = 0.$$

Solutions of these equations

$$K_2 = \pm(2p_1p_2u_1u_2 + 2v_1v_2f + v_1g_2 + v_2g_1) = \pm\frac{K_4 - g_1g_2}{2f} \tag{3.11}$$

have been found in [7] in framework of the Lagrangian formalism. The algebras of quadratic integrals of motion $H_{1,2}$ and K_2 have been considered in [1].

There is some opinion that all superintegrable systems with quadratic (linear) integrals of motion are multiseparable, i.e. allows the separation of variables in the Hamilton–Jacobi equation in at least two different coordinate systems on the configuration space [6]. Some of the Drach systems may be considered as counterexamples associated with the Lie surfaces [1].

Namely, three of the superintegrable Drach systems with quadratic integrals H_1, H_2, K_2 are separable in the one-coordinate system on the configuration space only.

Proposition 3. *For the (b) and (c) Drach systems, the integrals of motion H_1, K_2 are separable in the coordinates*

$$x = \frac{q_2}{2q_1}, \qquad y = q_1q_2,$$

and the corresponding separated relations do not allow us to get cubic integrals of motion.

For the (e) and (g) cases, the integrals of motion H_1, K_2 are separable in the coordinates

$$x = \frac{q_1 - q_2}{2}, \qquad y = -\frac{(q_1 + q_2)^2}{4}$$

and we can use the corresponding separated relations to the construction of the cubic integrals of motion.

For the (d), (f) and (k) cases, the quadratic integrals of motion H_1 and K_2 are not separable by the point transformations.

We can prove this proposition by using a computer program from [5].

3.4. The (l) system

Without loss of generality we can put $\rho = -3$ in the (l) case. Substituting this Hamiltonian into the computer program from [5], one gets the separated variables

$$x = \frac{(q_1 - q_2)^2}{2}, \qquad y = \frac{(q_1 + q_2)^2}{2}$$

and the corresponding separated relations

$$p_j^2 = P_j(q_j) = -4\alpha q_j^4 \mp 8\sqrt{2}\gamma q_j^3 + 4H_1 q_1^2 \mp 4\sqrt{2}\beta q_j + H_2, \qquad j = 1, 2, \tag{3.12}$$

which give rise to one hyperelliptic curve $\mu^2 = P(\lambda)$ at $\mu = p_j$ and $\lambda = \pm q_j$.

The angle variable

$$\begin{aligned}\omega_2 &= \frac{1}{2}\int^{q_1} \frac{\mathrm{d}\lambda}{\sqrt{P(\lambda)}} + \frac{1}{2}\int^{q_2} \frac{\mathrm{d}\lambda}{\sqrt{P(-\lambda)}} \\ &= \frac{1}{2}\int^{q_1} \frac{\mathrm{d}\lambda}{\sqrt{P(\lambda)}} - \frac{1}{2}\int^{-q_2} \frac{\mathrm{d}\lambda}{\sqrt{P(\lambda)}}\end{aligned} \tag{3.13}$$

is a sum of the incomplete elliptic integrals of the first kind on the common hyperelliptic curve. According to [2, 15] there is the addition theorem and an additional cubic integral of motion

$$K_3 = 2(\widetilde{P}_1' p_2 + \widetilde{P}_2' p_1),$$

which looks like as the Drach integral (3.6), but in this case functions

$$\widetilde{P}_{1,2}' = (q_1 + q_2)^2 \frac{\partial}{\partial q_{1,2}} \frac{P(\pm q_{1,2})}{(q_1 + q_2)^4}$$

have completely another algebro-geometric explanation. All the details will be published in the forthcoming publications.

As a sequence, the algebra of integrals of motion $H_{1,2}$ and K_3 differs from the corresponding algebras for other Drach systems related to another addition theorem. As an example, the recurrence chain $K_{j+1} = \{H_2, K_j\}$ terminates on the fourth step only

$$K_7 = \{H_2, K_6\} = -480K_4K_3 + 256H_1^2K_3 - 768\alpha H_2K_3 - 3072\beta\gamma K_3.$$

The solution of the inverse recurrence chain looks like

$$K_2 = p_y^2 + 2\alpha x - 4\gamma\sqrt{x}.$$

It is interesting that the algebra of quadratic integrals of motion $H_{1,2}$ and K_2 is one of the standard cubic algebras [1] and we do not provide the explanation of this fact.

4. New superintegrable systems on zero-genus hyperelliptic curves at $\kappa_1 = 1$ and $\kappa_2 = 2, 3$

Let us put $\kappa_1 = 1$ and $\kappa_2 = 2$ in (2.23) and try to solve equations (2.26) and (2.27). Here is one superintegrable system with cubic additional integral K_m (2.20) and quadratic integral K_ℓ (2.21)

$$V_{III} = \gamma_1(3x + y)(x + 3y) + \gamma_2(x + y) + \gamma_3(x - y), \qquad S = \begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix},$$

and seven systems with the real potentials

$$V_I = \gamma_1(3x+y)(x+3y) + \frac{\gamma_2}{(x+y)^2} + \frac{\gamma_3}{(x-y)^2}, \qquad S = \begin{pmatrix} a/q_1 & b/q_2 \\ 1/q_1 & 2/q_2 \end{pmatrix},$$

$$V_{II}^{(1)} = \gamma_1 xy + \frac{\gamma_2}{\sqrt{x^3y}} + \frac{\gamma_3}{x^2}, \qquad S = \begin{pmatrix} 1/q_1 & 1 \\ 1/q_1^2 & 4/q_2^2 \end{pmatrix},$$

$$V_{II}^{(2)} = \frac{\gamma_1}{\sqrt{xy}} + \frac{\gamma_2}{x^2} + \frac{\gamma_3 y}{x^3}, \qquad S = \begin{pmatrix} 0 & 1 \\ 1/q_1^2 & 4/q_2^2 \end{pmatrix},$$

$$V_{II}^{(3)} = \frac{\gamma_1}{\sqrt{xy}} + \frac{\gamma_2}{\sqrt{x^3y}} + \frac{\gamma_3}{x^{5/4}y^{3/4}}, \qquad S = \begin{pmatrix} 1 & 0 \\ 1/q_1^2 & 4/q_2^2 \end{pmatrix},$$

$$V_{II}^{(4)} = \gamma_1 xy + \frac{\gamma_2 y}{x^3} + \frac{\gamma_3 y^3}{x^5}, \qquad S = \begin{pmatrix} 0 & 4/q_2 \\ 1/q_1^2 & 4/q_2^2 \end{pmatrix},$$

$$V_{IV}^{(1)} = \frac{\gamma_1}{\sqrt{xy}} + \frac{\gamma_2(\sqrt{x}-\sqrt{y})}{\sqrt{xy}} + \frac{\gamma_3}{\sqrt{xy}(\sqrt{x}+\sqrt{y})^2}, \qquad S = \begin{pmatrix} 1 & q_2^2/4 \\ 1/q_1 & 1 \end{pmatrix},$$

$$V_{IV}^{(2)} = \gamma_1 xy + \gamma_2(x-y) + \frac{\gamma_3}{(x+y)^2}, \qquad S = \begin{pmatrix} a/q_1 & b \\ 1/q_1 & 1 \end{pmatrix},$$

for which the integrals of motion K_m and K_ℓ (2.20) and (2.21) are fifth- and sixth-order polynomials in the momenta.

The solution of the equations $K_m = \pm\{H_2, K_{m-1}\}$ and $\{H_1, K_{m-1}\} = 0$ looks like

$$\begin{aligned} K_{m-1} &= 4p_1p_2u_1u_2(2fv_2+g_2) + 4v_2(2fv_1+g_1)(fv_2+g_2) + \left(4fh_2+g_2^2\right)v_1 \\ &= 4\mu_2(\mu_1P_2' + P_1') - \left(4fh_2 - g_2^2\right)\lambda_1 - 4h_2g_1. \end{aligned}$$

It is an additional integral of motion, which is second-order polynomial for the system with potential V_{III} and fourth-order polynomial in the momenta for other systems.

Now we present some superintegrable Stäckel systems at $\kappa_1 = 1$ and $\kappa_2 = 3$. Here is one system with a cubic additional integral K_m (2.20)

$$V_{III}^{(1)} = \gamma_1(x+2y)(2x+y) + \gamma_2(x+2y) + \gamma_3(2x+y), \qquad S = \begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix},$$

and seven systems with the real potentials

$$V_I = \gamma_1(x+2y)(2x+y) + \frac{\gamma_2}{(x+y)^2} + \frac{\gamma_3}{(x-y)^2}, \qquad S = \begin{pmatrix} a/q_1 & b/q_2 \\ 1/q_1 & 3/q_2 \end{pmatrix},$$

$$V_{II}^{(1)} = \gamma_1 xy + \frac{\gamma_2}{x^{2/3}y^{4/3}} + \frac{\gamma_3}{x^{1/3}y^{5/3}}, \qquad S = \begin{pmatrix} 1/q_1 & 0 \\ 1/q_1^2 & 9/q_2^2 \end{pmatrix},$$

$$V_{II}^{(2)} = \frac{\gamma_1}{\sqrt{xy}} + \frac{\gamma_2\sqrt{y}}{x^{5/2}} + \frac{\gamma_3 y^2}{x^4}, \qquad S = \begin{pmatrix} 0 & 1 \\ 1/q_1^2 & 9/q_2^2 \end{pmatrix}$$

$$V_{II}^{(3)} = \frac{\gamma_1}{\sqrt{xy}} + \frac{\gamma_2}{x^{4/3}y^{2/3}} + \frac{\gamma_3}{x^{7/6}y^{5/6}}, \qquad S = \begin{pmatrix} 1 & 0 \\ 1/q_1^2 & 9/q_2^2 \end{pmatrix},$$

$$V_{II}^{(4)} = \gamma_1 xy + \frac{\gamma_2 y^2}{x^4} + \frac{\gamma_3 y^5}{x^7}, \qquad S = \begin{pmatrix} 0 & 1/q_2 \\ 1/q_1^2 & 9/q_2^2 \end{pmatrix},$$

$$V_{III}^{(2)} = \gamma_1(x^2 - 5x\sqrt{y} + 4y) + \frac{\gamma_2 x}{\sqrt{y}} + \frac{\gamma_3}{\sqrt{y}}, \qquad S = \begin{pmatrix} 2q_1 & 2q_2 \\ 1 & 1 \end{pmatrix},$$

$$V_{IV} = \gamma_1(x+5y)(5x+y) + \gamma_2(x-y) + \frac{\gamma_3}{(x-y)^2}, \qquad S = \begin{pmatrix} a/q_1 & b \\ 1/q_1 & 1 \end{pmatrix},$$

for which the integrals of motion K_m and K_ℓ (2.20) and (2.21) are seventh- and eighth-order polynomials in the momenta.

The algebra of integrals of motion $H_{1,2}$ and K_m (2.20) is the fifth- or seventh-order polynomial algebra in terms of the coefficients of the hyperelliptic curves

$$\{H_2, K_m\} = 2K_\ell, \qquad \{H_2, K_\ell\} = 2fK_m, \qquad \{K_m, K_\ell\} = \pm F_Z,$$

where the polynomial F_Z depends on the type of solution (2.24) only:

$$\begin{aligned}
F_I &= 2\big(4fh_2 - g_2^2\big)^{\kappa_2-\kappa_1}\big(4f\big(\kappa_1^2h_2g_1 + \kappa_2^2h_1g_2\big) - g_1g_2\big(\kappa_2^2g_1 + \kappa_1^2g_2\big)\big),\\
F_{II} &= 4\big(4fh_2 - g_2^2\big)^{\kappa_2-\kappa_1}\big(4f(\kappa_2+\kappa_1)h_2h_1 - \kappa_1h_1g_2^2 - \kappa_2h_2g_1^2\big) \mp K_m^2,\\
F_{III} &= 4\big(4fh_2 - g_2^2\big)^{\kappa_2-\kappa_1}\big(4f(\kappa_1h_2 + \kappa_2h_1) - \kappa_1g_2^2 - \kappa_2g_1^2\big)f,\\
F_{IV} &= 2\big(4fh_2 - g_2^2\big)^{\kappa_2-\kappa_1}\big(4f(2\kappa_2fh_1 - \kappa_1h_2g_1) - 2\kappa_2fg_1^2 + \kappa_1g_1g_2^2\big).
\end{aligned} \tag{4.1}$$

Here, the choice of sign + or − depends on κ.

As above, the Stäckel transformations (3.8) relate systems associated with one type of solutions (2.24), whereas the algebra of integrals of motion is invariant with respect to such transformations.

The complete classification of such superintegrable systems requires further investigations.

5. Conclusion

We discuss an application of the addition theorem to the construction of algebraic integrals of motion from the multi-valued action–angle variables.

We propose a new algorithm to the construction of the superintegrable Stäckel systems associated with zero-genus hyperelliptic curves. It allows us to prove that the Drach classification of the Stäckel systems with the cubic integral of motion (3.6) associated with the addition theorem (2.8) is complete. Moreover, we present some new two-dimensional superintegrable systems with third-, fifth- and seventh-order integrals of motion.

The proposed method may be applied to the construction of the higher order additional polynomial integrals of motion for the n-dimensional superintegrable Stäckel systems on the different manifolds. On the other hand, we prove that there are some superintegrable systems, which are missed out from this construction. It will be interesting to study a mathematical mechanism for the appearance of such superintegrable systems.

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